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INNOVATIVE PROCEDURE FOR THE ANALYSIS OF RELIABILITY DATA FROM TIME CENSORED SAMPLES

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ABSTRACT

Reliability experiments with time censoring are a common test practice for assessing the quality of products and/or processes. Parametric estimation is the target of every reliability data analysis. Maximum likelihood estimators are usually adopted to make the inference from time-censored samples but, except for the case of the exponential distribution, data analysis with the maximum likelihood principle only provides approximate confidence intervals. The use of central limit theorem allows to calculate confidence intervals that are exact only if the number of failures is infinite. Unfortunately, this is not the case of real practice: most reliability experiments test small samples (15-20 units) with very few failures (typically 6-7 units failed). In these situations, approximate confidence intervals, obtained by taking advantage of the good asymptotic properties of the maximum likelihood estimators, can give risky responses, i.e. confidence intervals that are narrower than the real ones.

The paper proposes an innovative procedure that allows to calculate always conservative approximate confidence intervals. By means of Monte Carlo simulations, the method, theoretically justified, can give approximate confidence intervals that are always larger than the real ones. Therefore, the procedure could give to technicians and engineers an useful tool to overcome the risks that are hidden behind the application of the standard asymptotic approximations. Finally, an example will show the applicability of the method; besides, comparisons with the results obtained making use of other non conservative approximations will show that the length of the confidence interval calculated with the new proposed approach is comparable with the other interval lengths.

Keywords: Axiomatic Design, Reliability, Confidence Intervals

1 INTRODUCTION

Assessing reliability improvements of products and processes is one of the main goals of axiomatic design. A designer needs to have an efficient and effective evaluation of the progresses that are occurring on the reliability of products or processes during the development of a project.

Suh [1] stated that in many situations it is not possible to establish tolerance on specific Functional Requirements; in these cases, if the FR is related to reliability, tolerance limits have to be evaluated via statistical analysis. Therefore, to assess either the reliability improvements or the tolerance limits, statistical tools are necessary to determine the confidence intervals of the design parameters, exactly or at least with a conservative approximation. If a design parameter is, for example, the life of a car and the designer wants to know if the parameter is higher than a specified value, he must determine the confidence interval for the life corresponding to a specified confidence level, with a good approximation. In these situations managers and designers must rely on statistical tools that give responses, if not exact, in any case conservative: i.e. approximate confidence intervals that are larger than the real ones.

Due to economical or physical constraints, reliability data are commonly subject to censoring: in some cases censoring is referred to the number of failures obtained during a test (failure censoring or type II censoring), while in some other cases censoring is related to the time of testing (time censoring or type I censoring). With time censoring, the number of failures R is a random variable, while, with failure censoring, r is a constant known by the experimenter. This distinction brings to different statistical analyses and different levels of approximation for the confidence intervals.

Log-location-scale distributions play an important role in reliability: Weibull, log-normal, as well as exponential are usual assumptions for samples obtained from survival experiments. Besides, because of their good asymptotic properties (consistency, unbiasedness, efficiency; see [2]), likelihood estimators, based on Maximum Likelihood Principle, are the most common estimators in parametric estimation.

Several authors studied the behavior of likelihood estimators subject to type II censoring: Lawless ([3], [4]), in particular, obtained exact inference procedures for log-location-scale distribution (i.e. exponential, Weibull and log-normal distributions), using a conditional approach; Billman et al. [5], considering type II censored samples from the Weibull distribution, gave tabulated values to adapt the distributions derived for the complete sample case, via Monte Carlo simulation, by Thoman et al. [6]. In case of type I censoring, except for the exponential distribution ([7], [8] and [9]), no exact procedure is available. Due to their computational heaviness, exact procedures are seldom used even when available, while asymptotic approximations are very spread; nevertheless, Piegorsch [10] suggested caution in the application of asymptotic methods in case of small samples arising from the two-parameter exponential distribution. Doganaksoy and Schmee [11], studying the accuracy of asymptotic inference procedures, pointed out that all the methods can be anticonservative (i.e. confidence intervals narrower than the real ones) even if sufficiently precise. Finally, Jeng and Meeker [12] and Wong and Wu [13] noted that bootstrap procedures can be more accurate but not always conservative.

This article provides a new method to obtain, by means of Monte Carlo simulations, always conservative confidence intervals in case of location-scale samples subject to simple type I censoring.

2 CONSERVATIVE APPROACH FOR TIME CENSORED SAMPLES

2.1 RELIABILITY MODELS

Location-scale models are quite common in lifetime data. The normal distribution and the smallest extreme value (SEV) distribution are the most famous examples of location-scale distributions. It can be demonstrated that if T is distributed like a Weibull, then Y=log(T) has a SEV distribution with parameters μ=log(η) and $\sigma=1/\beta$; besides, it is well known that if T is distributed like a log-normal distribution, then Y=log(T) has a normal distribution with the same parameters. If the random variable Y=log(T) is distributed like a location-scale model, then the random variable T has a log-location-scale distribution; therefore, two of the most used lifetime distributions (i.e., Weibull and log-normal) are log-location-scale distributions and inference on their parameters can be done studying the corresponding location-scale distributions (i.e., SEV and normal).

2.2 PARAMETER ESTIMATORS

The maximum likelihood estimators (MLE) can be obtained by maximizing the likelihood function for μ and σ simultaneously. In the exponential case the likelihood estimator of the scale parameter results $\hat{\sigma} = \sum_{i=1}^{n} T_i \Big/ R$, where R indicates the random variable that corresponds to the number of failures occurred in the test. It is obviously impossible to estimate the scale parameter without failures. This is true for all the location-scale models and for this reason the results reported in this paper will consider only the cases with R>0. In the Weibull (SEV) and log-normal (normal) cases an explicit expression for the MLE can not be obtained; it is necessary to solve a set of equations to get the estimates of the scale and location parameters (see [4]).

2.3 PARAMETER-FREE QUANTITIES

Random variables that do not depend on the unknown parameters are said pivotal or parameter-free. The distribution of a pivotal variable is the same for every choice of the parameters; it is, therefore, possible to obtain the distribution with samples simulated by imposing, for example μ=0 and σ=1 in case of location-scale models. Lawless ([3] and [4]) and Bain and Engelhardt [1] showed that with type II censoring or complete data and log-location-scale distributions, the four statistics $Z=(\hat{\mu}-\mu)/\sigma$, $Z_\mu=(\hat{\mu}-\mu)/\hat{\sigma}$, $Z_\sigma=\hat{\sigma}/\sigma$ and $Z_p=(\hat{\mu}-y_p)/\hat{\sigma}$ are pivotal quantities. With Z_μ and by means of Monte Carlo simulations (see [1], pp. 214-216), it is possible to make inference on the location parameter; with Z_σ, confidence intervals on σ are calculable; finally, Z_p allows to construct confidence intervals on the pth quantile. Another method that permits to calculate analytically the exact distribution for the three statistics is shown in Lawless [3]. It can be demonstrated (see Appendix) that, given the probability of failure corresponding to the censoring time t_c, with type I censoring the three statistics Z_μ, Z_σ and Z_p are still pivotal quantities.

2.4 CONSERVATIVE CONFIDENCE INTERVALS

To calculate the confidence intervals of the parameters, simulations were obtained in different conditions: samples from the exponential distribution of size n ranging from 2 to 50 and probability of failures P_f from .05 to 1 were in fact analyzed. In particular, n = 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20 and P_f = .05: .05: 1 (i.e., P_f varies from .05 to 1 with steps of .05) were considered. In all the simulations the scale parameter was set σ =1 and for each test-case 10000 simulated samples were generated using an algorithm coded in Matlab.

To obtain the confidence intervals of the location and scale parameters, it is necessary to evaluate, on the basis of the number of failures obtained in the simulation, the confidence interval for the probability of failures. The distribution of the number of failures R is well known to be a binomial with parameters n and P_f. For each combination of number of failure r and sample size n, it is possible to calculate a confidence interval for P_f, conditional on R>0. The confidence interval for P_f corresponding to the confidence level (1-α_p) is subdivided into n_{step} (we used n_{step}=1000) values and for each P_f an empirical distribution is obtained with n_{sim} (n_{sim}=5000 was used) simulations; the envelope of the calculated distributions gives the two limiting distributions that contain the real distribution with confidence level (1-α_p). From the envelope it is possible to calculate a conservative confidence interval (actually, this is a conditional confidence interval) for the pivotal quantities of interest, given that $P_{f,\inf} \le P_f \le P_{f,\sup}$: $P\left[z_{\inf} \le Z_{\mu,\sigma,p} \le z_{\sup} \middle| P_{f,\inf} \le P_f \le P_{f,\sup}\right] \ge 1-\alpha$. With simple passages it is easy to obtain, from the conditional probability written above, that the three confidence intervals $\hat{\mu}-z_{\mu,\sup}\hat{\sigma} \le \mu \le \hat{\mu}-z_{\mu,\inf}\hat{\sigma}$, $\hat{\sigma}/z_{\sigma,\sup} \le \sigma \le \hat{\sigma}/z_{\sigma,\inf}$ and $\hat{\mu}-z_{p,\sup}\hat{\sigma} \le y_p \le \hat{\mu}-z_{p,\inf}\hat{\sigma}$ result conservative (greater than the real confidence intervals) with respect to the theoretical confidence level $(1-\alpha_p)(1-\alpha)$.

Table 1 reports the 95% confidence intervals of the scale parameter for some test-cases (sample sizes 3, 5, 10, 16 and 20).

Number of failures	Sample size 3	Sample size 5	Sample size 10	Sample size 16	Sample size 20
1	.00 4.78	.00 5.05	.00 5.16	.00 5.23	.00 5.19
2	.03 4.78	.03 5.10	.03 5.41	.03 5.29	.03 5.33
3	.16 4.78	.15 5.10	.15 5.41	.14 5.29	.14 5.33
4	-	.25 5.10	.25 5.41	.24 5.29	.24 5.33
5	-	.30 5.10	.32 5.41	.31 5.29	.31 5.33
6	-	-	.36 5.41	.37 5.29	.37 5.33
7	-	-	.40 5.41	.41 5.29	.42 5.33
8	-	-	.42 5.41	.44 5.29	.44 5.33
9	-	-	.44 3.44	.46 3.49	.46 5.33
10	-	-	.46 2.39	.48 3.48	.48 3.51
11	-	-	-	.50 2.80	.50 2.88
12	-	-	-	.51 2.49	.52 2.88
13	-	-	-	.52 2.23	.53 2.48
14	-	-	-	.53 2.10	.54 2.25
15	-	-	-	.54 1.91	.55 2.09
16	-	-	-	.54 1.81	.56 1.97
17	-	-	-	-	.58 1.88
18	-	-	-	-	.58 1.81
19	-	-	-	-	.59 1.72
20	-	-	-	-	.59 1.66

Tab. 1 – Confidence intervals for Z_σ for different number of failures r and sample sizes 3, 5, 10, 16 and 20 (α = .05).

3 MONTE CARLO SIMULATION

3.1 ACTUAL CONFIDENCE LEVELS

Let (1-α_n) be the nominal confidence level of a confidence interval calculated for the scale parameter of an exponential distribution. The number of simulated intervals $N = \hat{\alpha} \cdot n_{sim}$ not including the true parameter is distributed like a binomial with parameters n_{sim} (10000 simulations were used) and α_n. The confidence interval for α_n corresponding to the 95% is, therefore, $\hat{\alpha} \pm 1.96\sqrt{\hat{\alpha}(1-\hat{\alpha})/n_{sim}}$. If the lower limit is higher than a confidence level α_n, then the method is said anticonservative with respect to α_n; while, if the nominal confidence level is included within the confidence interval or is higher than the upper limit, then the method is said conservative for α_n .

Figures 1-5 show the actual error percentage of the 95% confidence intervals for sample sizes 3, 5, 10, 16 and 20.

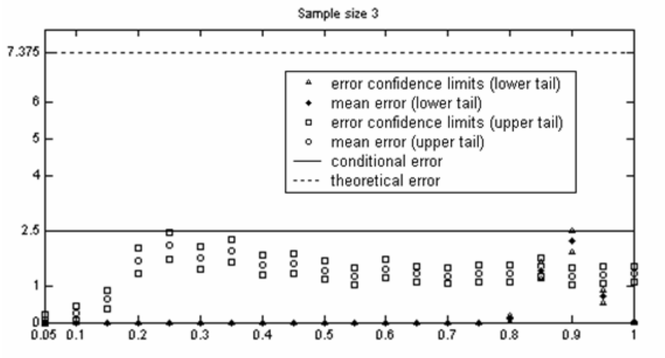

Figure 1 – Actual lower and upper percent error versus probability of failures plot of one-sided 97.5% confidence intervals for parameter σ. Sample size 3.

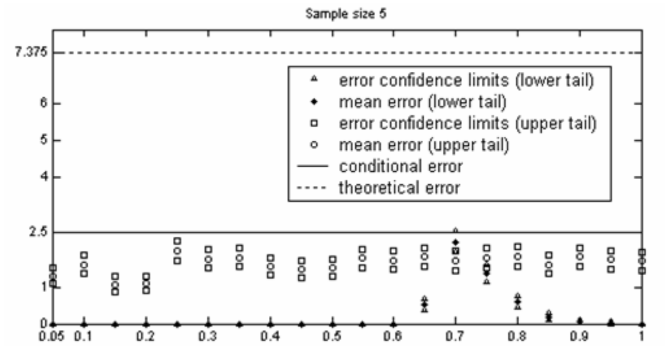

Figure 2 – Actual lower and upper percent error versus probability of failures plot of one-sided 97.5% confidence intervals for parameter σ. Sample size 5.

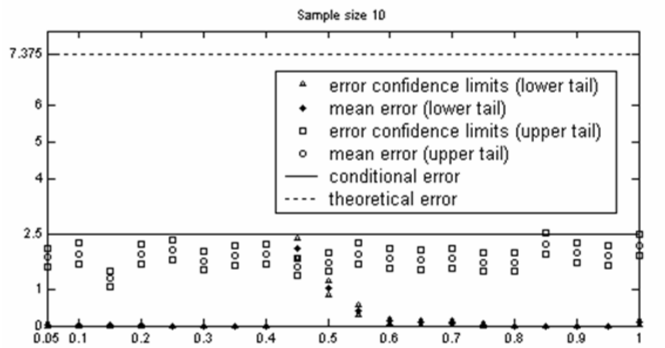

Figure 3 – Actual lower and upper percent error versus probability of failures plot of one-sided 97.5% confidence intervals for parameter σ. Sample size 10.

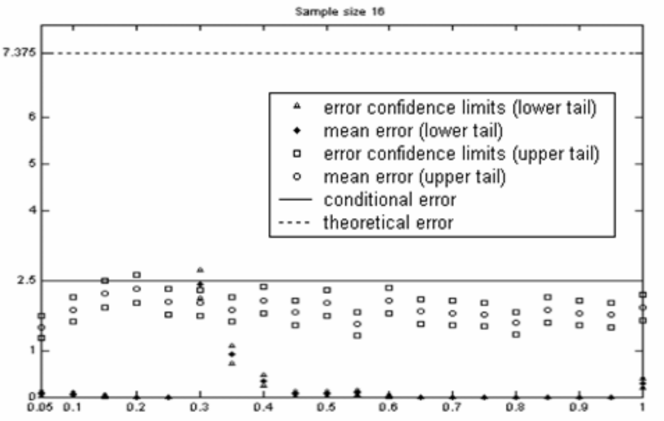

Figure 4 – Actual lower and upper percent error versus probability of failures plot of one-sided 97.5% confidence intervals for parameter σ. Sample size 16.

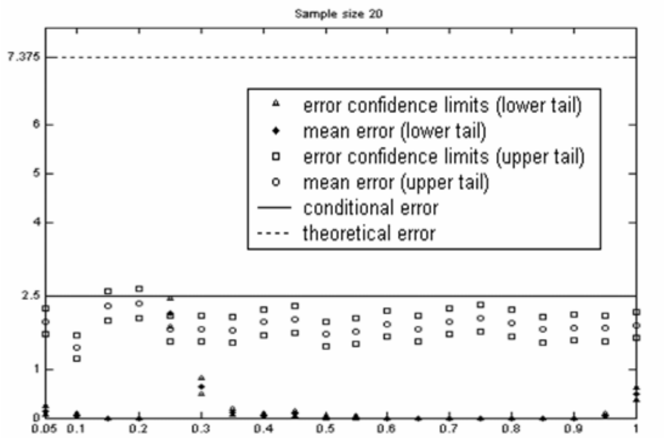

Figure 5 – Actual lower and upper percent error versus probability of failures plot of one-sided 97.5% confidence intervals for parameter σ. Sample size 20.

It is evident from the figures 1-5 that the intervals are conservative not only considering the product $(1-\alpha_p)(1-\alpha)$, but also for the conditional confidence level $(1-\alpha)$.

3.2 INTERVAL LENGTHS

Figure 6 shows the average interval lengths obtained simulating 5000 samples from the standard exponential distribution with probability of failures equal to .75.

Each curve is obtained calculating 95% confidence intervals with 7 different approximate methods: curve 1 derives from the classic asymptotic normal (AN) approximation (see [4]); curve 2 is determined considering the AN approximation proposed by Sprott [14]; curve 3 is obtained with the asymptotic LR approximation (see [4]); curve 4 is calculated with the χ^2 approximation proposed by Cox [15]; curve 5 is obtained with the approximate normal distribution of the random variable ξ, function of the MLE, proposed by Bartholomew [7]; curve 6 is obtained with confidence intervals calculated considering only the number of failures ([8]); finally, curve 7 is determined with the new conservative approach. As it can be seen from figure 6, pronounced differences exist only for sample size lower than 30. In any case, the new approach gives confidence intervals with lengths very similar to the ones obtained considering only the number of failures (curve 6). Besides, the new method and the method based on the number of failures give, as expected, the largest confidence intervals.

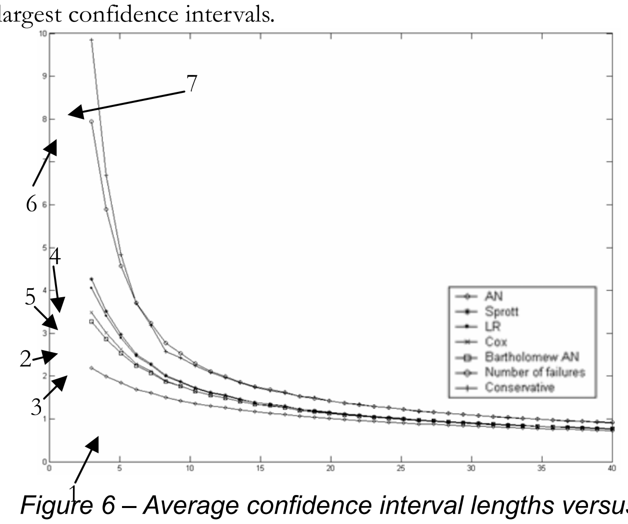

Figure 6 – Average confidence interval lengths versus sample size for the scale parameter of the exponential distribution. P_f = .75, α = .05 and sample sizes varying from 3 to 40 with increment 1 (n = 3:1:40).

4 NUMERICAL EXAMPLE

Differences among the methods can be shown with an example from Bartholomew [7]. The data arise from a life test in which 20 items, following the exponential distribution, are observed for 150 hours. During the observation period 15 items failed with failure times in hours: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109 and 138. The ML estimate for the mean time to failure is 105.8 hours. The lower and upper 95% confidence bounds for the scale parameter, calculated with the 7 different methods listed in section 3.2, are reported in table 2.

Curve	Method	Lower	Upper
1	AN	52.5	159.1
2	Sprott	66.4	183.6
3	LR	66.3	183.9
4	Cox	65.8	181.0
5	Bartholomew AN	68.8	184.6
6	N. of failures	61.3	210.9
7	Conservative	50.6	192.4

Tab. 2 – The lower and upper 97.5% confidence bounds for the scale parameter.

The limits for the conservative approach are obtained entering table 1 with r = 15 and n = 20 and dividing the mean time to failure by the tabulated values (i.e. *2.09* and *.55*). The example confirms the results obtained and discussed in section 3.2: except for the method 6, all the other methods give narrower confidence intervals with respect to the one calculated with the conservative approach (method 7). However, it should be noted that the lower and upper limits obtained with method 7 are similar, while for method 6 the lower limit is close to the other methods and the upper limit is definitely high (10% higher than the upper limit calculated with method 7).

5 CONCLUSIONS

In Axiomatic Design Functional Requirements and Design Parameters are often assessed by experimental testing; in such situations effective statistical methods are necessary to take the right decisions in terms of tolerance limits or product/process improvements. In particular, in the case of reliability requirements and parameters, it is essential to get at least conservative confidence intervals.
Methods to obtain exact confidence intervals for the parameters of the log-location-scale distributions are well known, while, to the authors' best knowledge, in case of type I censoring no method is available to determine conservative confidence intervals. This paper suggests a new method, which is theoretically justified, that allows to obtain conservative confidence intervals with time censoring. The article provides, as an application of the method, the study of exponential samples subject to type I censoring.
Simulations put in evidence that, by choosing a 95% confidence level for the probability of failure, the confidence intervals for the scale parameter of the exponential distribution are conservative not only for the theoretical limit $(1-.95)(1-\alpha)$, but also for the conditional confidence level $(1-\alpha)$. The same results have been obtained for the other log-location-scale distributions: 2-parameter exponential, Weibull and log-normal.
Some aspects, certainly, need further research. It would be important to analyze the influence of the confidence level $(1-\alpha_p)$ on the percent error: in particular, it would be interesting showing that a limiting value $(1-\alpha_p^*)$ for $(1-\alpha_p)$ exists, above which it is possible to obtain conservative confidence intervals with respect to the conditional confidence level $(1-\alpha)$; the analysis can be done by imposing increasing values of α_p, until obtaining confidence intervals anticonservative with respect to $(1-\alpha)$.

6 APPENDIX: INFERENCE FOR LOCATION-SCALE MODELS

Theorem:
Let Y be a random variable with density function $\varphi_Y(y;\mu,\sigma)=f((y-\mu)/\sigma)/\sigma$, where $\sigma>0$ and $\mu\in\Re$, and cumulative function $\Phi_Y(y;\mu,\sigma)=F((y-\mu)/\sigma)$.
In case of type I censoring or hybrid type I censoring, if the censoring value y^* gives $F((y^*-\mu)/\sigma)=cost$, for every μ and σ, the statistics, function of the MLE $\hat{\mu}$ and $\hat{\sigma}$, $Z=(\hat{\mu}-\mu)/\sigma$, $Z_\mu=(\hat{\mu}-\mu)/\hat{\sigma}$, $Z_\sigma=\hat{\sigma}/\sigma$ and $Z_p=(\hat{\mu}-y_p)/\hat{\sigma}$ are pivotal quantities.
Proof:
Given an hybrid type I censored sample (r^*, y^*) of size n, the likelihood results:

$$\begin{cases} L_{\mathbf{y}}(\mu,\sigma|\mathbf{y})=\dfrac{1}{\sigma^R}\left[\prod_{i=1}^{R} f\left(\dfrac{y_i-\mu}{\sigma}\right)\right]\cdot[k]^{n-R} \\ k=\begin{cases} 1-F\left(\dfrac{y_{r^*}-\mu}{\sigma}\right), & y_{r^*}\le y^* \\ 1-F\left(\dfrac{y^*-\mu}{\sigma}\right), & y_{r^*}>y^* \end{cases} \end{cases} \quad (1)$$

Considering the transformation $y'=cy+d$, $\mu'=c\mu+d$ and $\sigma'=c\sigma$ with c>0 and $d\in\Re$; according to the hypotheses, in order to obtain $F((y^{*\prime}-\mu')/\sigma')=F((y^*-\mu)/\sigma)$, $(y^*)'$ must be equal to cy^*+d.
The likelihood becomes with easy passages:

$$\begin{cases} L_{\mathbf{y}'}(\mu',\sigma'|\mathbf{y}')=\dfrac{1}{(c\sigma)^R}\left[\prod_{i=1}^{R} f\left(\dfrac{y_i-\mu}{\sigma}\right)\right]\cdot[k']^{n-R} \\ k'=\begin{cases} 1-F\left(\dfrac{y_{r^*}-\mu}{\sigma}\right), & y_{r^*}\le y^* \\ 1-F\left(\dfrac{y^*-\mu}{\sigma}\right), & y_{r^*}>y^* \end{cases} \end{cases} \quad (2)$$

From the relations (1) and (2), it follows:

$$L_{\mathbf{y}}(\mu,\sigma|\mathbf{y})=c^G L_{\mathbf{y}'}(\mu',\sigma'|\mathbf{y}') \quad (3)$$

Let $(\tilde{\mu},\tilde{\sigma})$ and $(\tilde{\mu}',\tilde{\sigma}')$ give the maximum of the functions $L_{\mathbf{y}}(\mu,\sigma|\mathbf{y})$ and $L_{\mathbf{y}'}(\mu',\sigma'|\mathbf{y}')$, from (3) it follows, considering the estimators:

$$\begin{cases} \hat{\mu}(\mathbf{Y}')=c\hat{\mu}(\mathbf{Y})+d \\ \hat{\sigma}(\mathbf{Y}')=c\hat{\sigma}(\mathbf{Y}) \end{cases} \quad (4)$$

Since $X=(Y-\mu)/\sigma$ has probability density $\varphi_W(x;\mu,\sigma)$ not depending on μ and σ, from (4), it follows that the statistics $\hat{\mu}(\mathbf{X})=\hat{\mu}(\mathbf{Y}/\sigma-\mu/\sigma)=(\hat{\mu}(\mathbf{Y})-\mu)/\sigma=Z$ and $\hat{\sigma}(\mathbf{X})=\hat{\sigma}(\mathbf{Y}/\sigma)=\hat{\sigma}(\mathbf{Y})/\sigma=Z_\sigma$ are pivotal quantities.
Besides, since $Z_\mu=Z/Z_\sigma$ and $Z_p=y_p/Z_\mu-Z_\sigma$, also Z_μ and Z_p do not depend on μ and σ.
The same passages can be repeated in the case of type I censored samples, setting $k=1-F((y^*-\mu)/\sigma)$.

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